

TWO-FOLD SYMMETRIC SINGULARITY

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ABSTRACT. We explore some qualitative dynamics in the neighborhood of the 3 – *dimensional* two-fold symmetric singularity. We study the existence of an one-parameter family of regular (pseudo) periodic orbits of such systems near a reversible two-fold singularity.

1. INTRODUCTION

Theory of non-smooth dynamical systems has been developing at a very fast pace in recent years and it has become certainly one of the common frontiers between Mathematics and Physics or Engineering. Hybrid and switched models are being increasingly used in applications to describe a large variety of physical devices. Examples include mechanical systems with friction and backlash, electrical and electronic circuits, walking and hopping robots and, more recently, biological and neural systems [B-S-C, C, K-R-G]. One should observe that much work has been done in the study of the qualitative aspects of the phase space of discontinuous vector fields.

We consider vector fields expressible in the form $\dot{x} = Z(x)$ where $x \in \mathbb{R}^3$ is a state vector and Z is a smooth piecewise mapping. The discontinuities are concentrated on a codimension-one submanifold Σ of \mathbb{R}^3 . Σ is usually called the switching manifold. Orbit-solutions on Σ , whenever possible are defined according to the Filippov convention, [F]. We point out that trajectories may become constrained to the switching manifold, and this behavior is called sliding.

A general understanding of dynamics of generic 3D Filippov systems was obstructed by the appearance of the *two-fold singularity* (refer to 2.2 and [T1]). Observe that, topologically speaking, its shape is very simple, and moreover it is generic in piecewise smooth systems with three or more dimensions.

In the last years we have been noticing a great interest in the study of the two-fold singularity (also called T-singularity). In [J-C] is classified the dynamics in a neighborhood of a T-singularity. In [C-B-F-J] her occurrence was discussed with real models in control theory, introducing conditions for the study of the existence of this singularity.

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Let us consider a discontinuous vector field $Z = (X, Y)$ and its corresponding *normalized sliding vector field* \tilde{Z}^S (see Definition 1 and Remark 1). Let p_0 a point where Z presents a discontinuity. When p_0 is the two-fold singularity, p_0 is always a critical point of \tilde{Z}^S and a fixed point of the *return map* (see definition 4) which is a composition $\varphi_Z = \gamma_Y \circ \gamma_X$ of two involutions associated respectively to Y and X .

We deal with perturbations Z (whose expressions will be precise in 3.1) of the 3D piecewise smooth vector fields $Z_0 = (X_0, Y_0)$. Our aim is to find conditions to the existence of typical closed orbits of Z (see definition 9). To do this we use some geometrical properties, like a reversibility. So, here we exhibited a subset of this typical singularities that is reversible.

The paper is organized as follows: in sections 2 we deal with some preliminaries, give some definitions, and establish the notations. In section 3 we state the results about the reversibility. In section 3 and 4 we prove the existence of a family of the regular and pseudo periodic orbits, respectively.

2. PRELIMINARIES

2.1. Distinguished regions of the discontinuity set. In this section some notations, basic definitions and elementary concepts are presented.

Designate by \mathfrak{X}^r the space of all germs of C^r vector fields on \mathbb{R}^3 at 0 endowed with the C^r -topology with $r > 1$ and large enough for our purposes. Call Ω^r the space of all germs of vector fields Z in $(\mathbb{R}^3, 0)$ such that

$$Z(q) = \begin{cases} X(q), & \text{for } h(q) > 0, \\ Y(q), & \text{for } h(q) < 0, \end{cases}$$

where h is (a germ of) a smooth function $h : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}, 0)$ having 0 $\in \mathbb{R}$ as its regular value. Let $\Sigma = h^{-1}(0)$. We assume that $0 \in \Sigma$.

The above vector field is denoted by $Z = (X, Y)$. We endow $\Omega^r = \mathfrak{X}^r \times \mathfrak{X}^r$ with the product topology.

To define the orbit solutions of Z on the switching surface Σ we follow a pragmatic approach. In a well characterized open set \mathcal{O} of Σ (described below) the solution of Z through a point $p \in \mathcal{O}$ obeys the Filippov rules (see [F]) and on $\Sigma - \mathcal{O}$ we accept it to be multivalued. As we are dealing with systems derived from ordinary differential equations the non-uniqueness of solutions is allowed. We just must take into account all the leaves of the foliation in \mathbb{R}^3 generated by the orbits of Z (and also the orbits of X and Y) passing through or exiting from or converging to a point $p \in \Sigma$.

For each $X \in \mathfrak{X}^r$ we define the smooth function $Xh : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $Xh = X \cdot \nabla h$ where \cdot is the canonical scalar product in \mathbb{R}^3 . In what follows we use the Filippov convention (refer to [F]). We first distinguish the following regions on Σ :

- **Sewing Region:** $SwR = \{p \in \Sigma; (Xh)(p)(Yh)(p) > 0\}$. When convenient we denote $SwR^+ = \{p \in \Sigma; (Xh)(p) > 0, (Yh)(p) > 0\}$ and $SwR^- = \{p \in \Sigma; (Xh)(p) < 0, (Yh)(p) < 0\}$. In general a point

in the phase space which moves on an orbit of Z and reaches a point in SwR , crosses Σ .

- **Escaping Region:** $EscR = \{p \in \Sigma; (Xh)(p) > 0, (Yh)(p) < 0\}$. In this case any orbit which meets $EscR$ remains tangent to Σ for negative times.
- **Sliding Region:** $SlR = \{p \in \Sigma; (Xh)(p) < 0, (Yh)(p) > 0\}$. On SlR the flow slides on Σ ; the flow follows a well defined vector field Z^S called the *sliding vector field* (see Definition 1).

Generically, the set $\mathcal{O} = SlR \cup EscR \cup SwR$ is open and dense in Σ . Observe that for any $p \in \mathcal{O}$, we have $X(p) \neq 0$ and $Y(p) \neq 0$.

Definition 1. Let $Z = (X, Y) \in \Omega^r$. The sliding vector field Z^S associated to Z is a linear convex combination of X and Y tangent to Σ , that is,

$$Z^S = \frac{1}{(Y - X) \cdot \nabla h} (Y \cdot \nabla h X - X \cdot \nabla h Y).$$

Remark 1. Observe that $EscR$ for Z represent SlR for $-Z$. We can define the escaping vector field on Σ by $-(-Z)^S$. This vector field is called the sliding vector field independently of whether it is defined in the sliding or escaping region. For $p \in SlR \cup EscR$ the local orbit of p is ruled by this vector field. Therefore all this orbit is contained in $SlR \cup EscR$, and the future orbit of the sliding vector field coincides with the future orbit of the normalized sliding vector field $\tilde{Z}^S = (Y \cdot \nabla h X - X \cdot \nabla h Y)$.

Observe that Z^S and \tilde{Z}^S are orbitally equivalent on SlR (resp. on $EscR$). Moreover \tilde{Z}^S can be C^r -extended beyond the boundary of $SlR, EscR$. For technical reasons we consider the future orbit of Z through a point $p \in SlR \cup EscR$ given by the orbit of \tilde{Z}^S .

Notation: In all what follows we consider the map $h : (x, y, z) \mapsto z$. So, the expression of \tilde{Z}^S is:

$$\tilde{Z}^S = (X^1 Y^3 - Y^1 X^3, X^2 Y^3 - Y^2 X^3),$$

where $X = (X^1, X^2, X^3)$ and $Y = (Y^1, Y^2, Y^3)$.

Definition 2. We say that 0 is a two-fold singularity of $Z = (X, Y) \in \Omega^r$ if $Xh(0) = Yh(0) = 0$ and $X^2h(0) \neq 0, Y^2h(0) \neq 0$, where $X^2h(0) = X(Xh)(0)$.

Definition 3. If $p \in SlR \cup EscR$ and $X(p), Y(p)$ are linearly dependent then p is a critical point of Z^S . In this case p is called a pseudo equilibrium of Z .

The curves of *tangential singularities* or the Σ -singularity of X in Σ are given by $S_X = \{p \in \Sigma; Xh(p) = 0\}$.

We denote by $\Omega^F \subset \Omega^r$ the set of non-smooth vector fields such that the origin is a two-fold singularity. First of all, observe that:

- 1- The trajectories of both X and Y through 0 have a quadratic contact with Σ (at 0).
- 2- Generically, S_X and S_Y are transverse at 0. This case was first studied in [T1].
- 3- If 0 is a two-fold singularity of Z then \tilde{Z}^S can be C^r -extended to a full neighborhood of $0 \in \Sigma$, $\tilde{Z}^S(0) = 0$ and $\varphi_Z(0) = 0$.

2.2. Two-fold singularity. The following construction is given in [T2]. Let $Z = (X, Y) \in \Omega^r$ such that 0 is a fold point for *both* X and Y . Applying the Implicit Function Theorem, for each $p \in (\Sigma, 0)$ there exists a unique $t(p)$ such that the orbit-solution $t \mapsto \phi_X(t, p)$ of X through p meets Σ at a point $\tilde{p} = \phi_X(t(p), p)$.

We then define the smooth mapping $\gamma_X : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ by $\gamma_X(p) = \tilde{p}$. This map is a C^r -diffeomorphism and satisfies: $\gamma_X^2 = Id$. Analogously, we define the smooth map associated to Y : $\gamma_Y : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ which satisfies $\gamma_Y^2 = Id$. We define now the first return map associated to $Z = (X, Y)$:

Definition 4. *The first return map $\varphi_Z : (\Sigma, 0) \rightarrow (\Sigma, 0)$ is defined by the composition*

$$\varphi_Z = \gamma_Y \circ \gamma_X$$

We denote by $L_Z(., .)$ the linear part of φ_Z .

φ_Z is a C^r -diffeomorphism preserving area: $\det(D\varphi_Z)(0, 0) = 1$. So the eigenvalues of $D\varphi(0, 0)$ are β and β^{-1} with:

- (a) saddle type: $\beta \in \mathbb{R}$, $\beta \neq 0$;
- (b) elliptical type: $\beta = e^{i\theta}$ with $\theta \in]0, \pi[$. In this case, $L_Z(., .)$ is a rotation.

2.3. Partition of Ω^F . Consider the subset of Ω^F :

Regular two-fold: Let $\Omega_0^F(\delta)$ be the set of all $Z = (X, Y) \in \Omega^F$ such that the contact between S_X and S_Y at 0 is transverse, the eigenvectors of $D\tilde{Z}^S(0)$ are transverse to S_X and S_Y at 0 and 0 is a hyperbolic critical point for \tilde{Z}^S .

Remark 2. *From [S-T] we derive that $\Omega_0^F(\delta)$ is a codimension zero submanifold of Ω .*

In $\Omega_0^F(\delta)$ we distinguish the following subsets, corresponding to three cases:

Elliptic case: $\Omega_0^F(\delta.1) = \{Z \in \Omega_0^F(\delta); X^2h(0) < 0 \text{ and } Y^2h(0) > 0\}$. We have two invisible tangencies (*invisible two-fold* or a *T-singularity*). See Figure 1;

Parabolic case: $\Omega_0^F(\delta.2) = \{Z \in \Omega_0^F(\delta); X^2h(0) > 0, Y^2h(0) > 0 \text{ or } X^2h(0) < 0, Y^2h(0) < 0\}$ (visible fold- invisible fold);

Hyperbolic case: $\Omega_0^F(\delta.3) = \{Z \in \Omega_0^F(\delta); X^2h(0) > 0, Y^2h(0) < 0\}$ (visible two-fold).

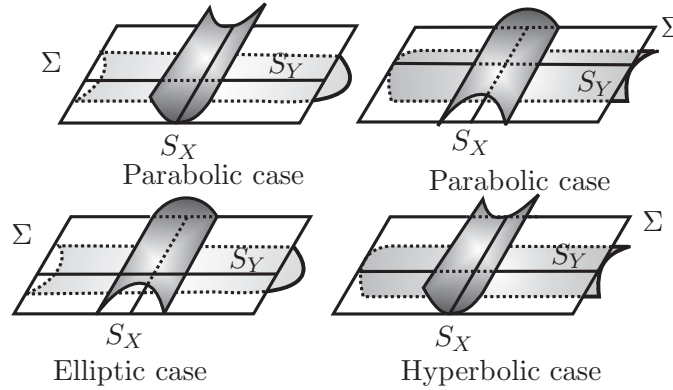


FIGURE 1. Two-fold singularity.

Recently, many works tried to understand the dynamics of the T-singularity, [J-C, J, C-B-F-J]. See [T2] for further references and related topics.

Definition 5. We say that a C^∞ -diffeomorphism $\xi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, is an involution if $\xi \circ \xi = Id$. We put $Fix(\xi) = \{p \in \mathbb{R}^4 \mid \xi(p) = p\}$.

Definition 6. We say that a non-smooth vector field $Z \in \Omega^r$ with discontinuity manifold Σ is ξ -reversible if there exists an involution $\xi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that:

- 1- $Fix(\xi) \subset \Sigma$
- 2- $\forall p \notin \Sigma, \xi \circ Z(p) = -Z(\xi(p))$.

Definition 7. Let $\Omega_0^\xi(\delta)$ be the set of elements $Z \in \Omega_0^F(\delta)$ such that:

- 1- Z is ξ -reversible;
- 2- Z is a generic invisible two-fold at 0;

Definition 8. We say that $Z \in \Omega^r$ is simple at 0 if $X^2h(0) \neq \pm XYh(0)$.

We say that $X \in \mathfrak{X}^r$ is semi-linear if $X^1(x, y, z), X^2(x, y, z)$ possess only the 0-jet (degree zero) and $X^3(x, y, z)$ possess only the 1-jet (degree one). Put:

$$SL(\Omega_0^F(\delta)) = \{Z = (X, Y) \in \Omega_0^F(\delta); X \text{ and } Y \text{ are semi-linear}\}.$$

Consider the set of non smooth dynamics vector fields:

$$\Omega_0^P(\delta) = \{Z \in SL(\Omega_0^F(\delta)); X^2h(0) < 0, (X(Yh))(0) < 0,$$

$$(X(Yh))(0)(Y(Xh))(0) = (X^2h)(0)(Y^2h)(0), Y^2h(0) > 0\}.$$

2.4. Orbits. By convention, if $p \in SLR$ the future orbit of Z through p is given by the trajectory of the sliding vector field Z^S through p .

Definition 9. Let $Z = (X, Y) \in \Omega^r$ and $x_0 \in (\mathbb{R}^3, 0)$.

- 1- The orbit $t \mapsto \phi_Z(t, x_0)$ is a regular periodic orbit if it is closed and composed by segments of orbits of X, Y and Z^S keeping the orientation. See Figure 2.
- 2- We call pseudo periodic regular orbit a trajectory of Z which is closed, composed by segments of the orbits of X and Y , with non preserved orientation (see Figure 2).

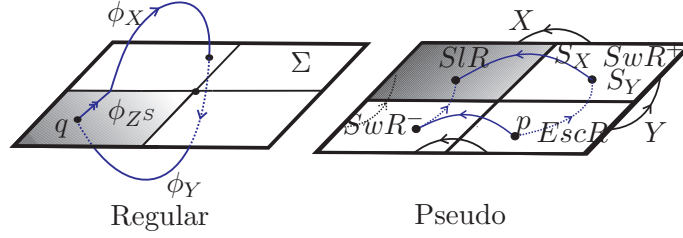


FIGURE 2. Periodic orbits.

2.5. Main results. Consider the set of straight line passing for the origin:

$$\mathcal{C}_\alpha = \{(x, \alpha x, 0) \in \Sigma; \text{ with } \alpha \in \mathbb{R}\}.$$

We obtain the result about the existence of pseudo periodic orbits and periodic orbits for $Z \in \Omega_0^\xi(\delta)$:

Theorem 10. *Let $Z \in \Omega_0^\xi(\delta)$ and $p \in \mathcal{C}_\alpha$. Then:*

- (a) *If Z is simple at 0 then Z does not have k -periodic orbits for all $k \in \mathbb{N}$.*
- (b) *If Z is non simple at 0 and $\alpha = 1$ then Z has one-periodic orbit passing through p . In addition, for $\alpha \neq 1$ does not exist k -periodic orbits for all $k \in \mathbb{N}$.*
- (c) *If Z is non simple at 0 and $\alpha = -1$ then Z has one-pseudo periodic orbit passing through p . In addition, for $\alpha \neq -1$ does not exist k -pseudo periodic orbits for all $k \in \mathbb{N}$.*
- (d) *0 is a hyperbolic equilibrium saddle (elliptical) point of φ_Z provided that $|XYh(0)| > |X^2h(0)|, (|XYh(0)| < |X^2h(0)|)$, see Figure 3.*

In Theorem 10 we exhibit conditions on $Z \in \Omega_0^\xi(\delta)$ for the existence of one-parameter families of periodic and pseudo periodic orbits for Z .

We also study the existence of pseudo periodic orbits and regular periodic orbits for $Z \in SL(\Omega_0^F(\delta))$. Observe that hypothesis in Theorem 11 (a) complete the result obtained in Theorem 10 (c) for $Z \in SL(\Omega_0^F(\delta))$. In the item (b) we get a more general family of regular periodic orbits, generalizing the family obtained in Theorem 10 (b), for $Z \in SL(\Omega_0^F(\delta))$. There results are summarized in:

Theorem 11. *Consider $Z = (X, Y) \in SL(\Omega_0^F(\delta))$, $p \in \mathcal{C}_\alpha$ and $X^2h(0) < 0, Y^2h(0) > 0$. Then*

- (a) *If Z is simple at the origin, $XYh(0) = -YXh(0)$, $X^2h(0) = -Y^2h(0)$ and $\alpha = -1$ then Z has a 1-pseudo periodic orbit passing through p . In addition, for $\alpha \neq -1$ there does not exist k -pseudo periodic orbits for all $k \in \mathbb{N}$.*
- (b) *If $Z \in \Omega_0^P(\delta)$, $\alpha = \lambda^*$ and $p \in SwR$ then p is either a fixed point for φ_1 or φ_2 , where $\lambda^* = \frac{X^2h(0)}{XYh(0)}$.*

3. REVERSIBILITY

In this section we will explore some symmetry properties of $Z \in \Omega_0^F(\delta)$. Our objective is to study the existence of periodic orbits of arbitrary period k .

3.1. Normal forms for the reversible two-fold singularity. Consider $Z_0 \in \Omega_0^F(\delta)$. Throughout this section, we fix a coordinates system (x, y, z) such that

$$\xi(x, y, z) = (y, x, -z).$$

Observe that $Fix(\xi) = \{(x, y, z); x = y, z = 0\}$.

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a non-zero polynomial in (x, y, z) . We call $d_T(f)$ the *total degree* of f , that is the maximum sum of exponents of the monomials of f .

Consider the polynomials: $f_\sigma^i : \mathbb{R}^3 \rightarrow \mathbb{R}$ where $\sigma \in \{x, y\}$ and $i = 1, 2, 3$, with the following specifications: $f_\sigma^{1,2}$ have null linear part and $d_T(f_\sigma^3) > 1$. Let

$$F_\sigma(x, y, z) = (f_\sigma^1(x, y, z), f_\sigma^2(x, y, z), f_\sigma^3(x, y, z)).$$

In what follows we exhibit the topological normal form for the two-fold singularity:

Proposition 12. (Normal Forms) *If $Z \in \Omega_0^F(\delta)$ then Z is C^0 -equivalent to $\tilde{Z} = (\tilde{X}, \tilde{Y})$ where*

$$\begin{aligned} \tilde{X}(x, y, z) &= (C_X, C_{XY}, x) + F_x(x, y, z), \\ \tilde{Y}(x, y, z) &= (C_{YX}, C_Y, y) + F_y(x, y, z), \end{aligned}$$

with $C_X = X^2h(0), C_Y = Y^2h(0), C_{XY} = X(Yh)(0), C_{YX} = Y(Xh)(0), X^2h(0) \neq 0$ and $Y^2h(0) \neq 0$.

Proof. The origin belongs to the frontier of the sliding region. Consider a local coordinates system such that $S_X = \{(x, y, z); x = z = 0\}$ and $S_Y = \{(x, y, z); y = z = 0\}$. With these settings the local normal forms of X and Y are:

$$X(x, y, z) = (C_X, C_{XY}, x) + F_x(x, y, z)$$

$$Y(x, y, z) = (C_{YX}, C_Y, y) + F_y(x, y, z).$$

Observe that $X^2h(0) = C_X, Y^2h(0) = C_Y, C_{XY} = X(Yh)(0)$ and $C_{YX} = Y(Xh)(0)$. ■

Consider the regions in Σ :

$$(1) \quad \begin{aligned} SlR &= \{(x, y, 0); x < 0, y > 0\}, \\ SwR &= \{(x, y, 0); x, y > 0\}, \\ EscR &= \{(x, y, 0); x > 0, y < 0\}. \end{aligned}$$

3.2. Dynamic of φ_Z when Z is ξ -reversible. The next Lemma exhibit the subset of $\Omega_0^F(\delta)$ that is ξ -reversible.

Lemma 13. *Let $Z \in \Omega_0^F(\delta)$. If $XYh(0) = -YXh(0)$, $X^2h(0) = -Y^2h(0)$ then Z is ξ -reversible.*

Proof. Straight forward computations. ■

The proof of Theorem 10 item (d) follows by:

Lemma 14. *Let $Z = (X, Y) \in \Omega_0^\xi(\delta)$. Then:*

- (i) *0 is a hyperbolic equilibrium saddle point of φ_Z provided that $|XYh(0)| > |X^2h(0)|$, see Figure 3;*
- (ii) *0 is a hyperbolic equilibrium elliptical point of φ_Z provided that $|XYh(0)| < |X^2h(0)|$.*

Proof. Consider the local normal form of $Z = (X, Y)$ given in the Proposition 12. By the Lemma 13 we have: $XYh(0) = C_{XY}, YXh(0) = C_{YX}, X^2h(0) = C_X$ and $Y^2h(0) = C_Y$.

The regions in Σ are given in (1). Let $p_0 = (x_0, y_0, z_0)$. The flows of the vector fields X and Y are:

$$(2) \quad \begin{aligned} \phi_X^t(p_0) &= (x_0 + C_X t, y_0 + C_{XY} t, z_0 + x_0 t + \frac{1}{2} C_X t^2) + O(t^2, t^2, t^3) \\ \phi_Y^t(p_0) &= (x_0 + C_{YX} t, y_0 + C_Y t, z_0 + y_0 t + \frac{1}{2} C_Y t^2) + O(t^2, t^2, t^3). \end{aligned}$$

If $p_0 = (x_0, y_0, 0) \in \Sigma$, taking

$$t_1(p_0) = -\frac{2x_0}{C_X} \quad \text{and} \quad t_2(p_0) = \frac{-4C_{XY}x_0 + 2C_X y_0}{C_X^2}$$

we obtain $\phi_X^{t_1}(p_0) = (x_1, y_1, 0) \in \Sigma$ and $\phi_Y^{t_2}(x_1, y_1, 0) \in \Sigma$. We define the return region (the shaded region in the Figure 3) by:

$$R_0 = [\{(x, y, 0) \in \Sigma; x > 0, y < 2C_{XY}C_X^{-1}x\} \cap SwR].$$

The return region R_0 represent the points $p = (x, y, 0) \in \Sigma$ such that $t_i(p), i = 1, 2$ are positive. The first return map is expressed by:

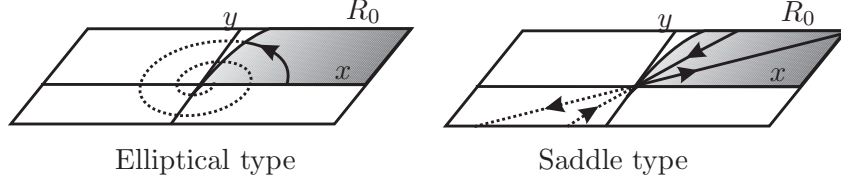


FIGURE 3. Dynamics of φ_Z .

$$(3) \quad \varphi_Z(p) = \phi_Y^{t_2} \circ \phi_X^{t_1}(p) = \left(\left(\frac{4C_{XY}^2 - C_X^2}{C_X^2} \right) x - \frac{2C_{XY}}{C_X} y, \frac{2C_{XY}}{C_X} x - y \right).$$

The eigenvalues of φ_Z are $\lambda_{\pm} = -1 + \frac{2(C_{XY}^2 \pm \sqrt{C_{XY}^2(C_{XY} - C_X)(C_{XY} + C_X)})}{C_X^2}$. Besides, the origin is a critical point of the saddle (respectively elliptical) type of φ_Z if $|C_{XY}| > |C_X|$ (respectively $|C_{XY}| < |C_X|$).

■

3.3. Existence of k -periodic orbits for Z . We investigate the existence of periodic orbits of $Z \in \Omega_0^\xi(\delta)$ of period k arbitrary. We prove the items (a), (b) and (c) of Theorem 10:

Proof. By (3), the equation $\varphi_Z(x, y, 0) = (x, y, 0)$ is expressed by:

$$\begin{cases} \frac{-C_X^2 + 4C_{XY}^2}{C_X^2} x - \frac{2C_{XY}}{C_X} y = x \\ \frac{2C_{XY}}{C_X} x - y = y. \end{cases}$$

The solutions are $C_{XY} = \pm C_X$. So, the periodic orbits pass through the pair of straight lines $\{(x, y); x = \pm y\}$. Observe that the straight line

$$\mathcal{C}_{-1} = \{(x, y, 0); x = -y\}$$

is contained in $SlR \cup EscR$. So we get a one-parameter family of pseudo periodic orbits.

On the straight line

$$\mathcal{C}_1 = \{(x, y, 0); x = y\} = Fix(\xi)$$

we obtain a one-parameter family of periodic orbits. The expression of φ_Z^n is given by:

$$\begin{aligned} \psi^n(x, \alpha x) = \varphi_Z^n(x, \alpha x) &= \left(\left[\frac{-C_X^2 + 4C_{XY}^2}{C_X^2} - \frac{2\alpha C_{XY}}{C_X} \right]^n x, \right. \\ &\quad \left. \left[\frac{-C_X^2 + 4C_{XY}^2}{C_X^2} - \frac{2\alpha C_{XY}}{C_X} \right]^{n-1} \left[\frac{2C_{XY}}{C_X} - \alpha \right] x \right). \end{aligned}$$

Solving $\psi^n(x, \alpha x) = (x, \alpha x)$, we obtain the system

$$\begin{cases} \left[\frac{-C_X^2 + 4C_{XY}^2}{C_X^2} - \frac{2\alpha C_{XY}}{C_X} \right]^n = 1 \\ \left[\frac{-C_X^2 + 4C_{XY}^2}{C_X^2} - \frac{2\alpha C_{XY}}{C_X} \right]^{n-1} \left[\frac{2C_{XY}}{C_X} - \alpha \right] = \alpha. \end{cases}$$

We obtain directly that $C_X = \frac{2\alpha C_{XY}}{1+\alpha^2}$. Replacing this value in the first equation, we obtain $\alpha^{-2n} = 1$, ie, $\alpha = \pm 1$. So the conclusion of the Theorem 10 is straight forward. ■

4. PSEUDO PERIODIC ORBITS

4.1. Existence of k -pseudo periodic orbits for Z . A necessary condition for the existence of pseudo periodic orbits is

$$\varphi_1(x, y) = \varphi_2(x, y)$$

where $\varphi_1(x, y) = (\phi_Y^{t_2} \circ \phi_X^{t_1})(x, y, 0)$ and $\varphi_2(x, y) = (\phi_X^{t_4} \circ \phi_Y^{t_3})(x, y, 0)$, with $t_i \geq 0$, for $i = 1, \dots, 4$. See Figure 4.

The regions on Σ are given in (1). We define the subset of $EscR \subset \Sigma$:

$$RS = \left\{ (x, y, 0) \in EscR; \frac{2Y(Xh)(0)}{Y^2h(0)}y < x < \frac{X^2h(0)}{2X(Yh)(0)}y \right\}.$$

Lemma 15. *Let $Z \in SL(\Omega_0^F(\delta))$. If $X^2h(0) < 0, Y^2h(0) > 0, XYh(0) = -YXh(0), X^2h(0) = -Y^2h(0)$ and $XYh(0)YXh(0) = X^2h(0)Y^2h(0)$ then $RS = \emptyset$.*

Proof. In fact,

$$\frac{2YXh(0)}{Y^2h(0)}y < \frac{X^2h(0)}{2XYh(0)}y \iff 4X^2h(0)Y^2h(0) > X^2h(0)Y^2h(0).$$

But this is a contradiction since $X^2h(0) < 0$ and $Y^2h(0) > 0$, by hypothesis. ■

We recall the notations:

$$\mathcal{C}_\alpha = \{(x, \alpha x, 0) \in \Sigma; \text{ with } \alpha \in \mathbb{R}\}.$$

Now we conclude the proof of Theorem 11:

Proof of Theorem 11: Consider the regions in Σ given by (1). The flow given in (2). So, the expressions of $\varphi_1(x, y)$ and $\varphi_2(x, y)$ are:

$$(4) \quad \begin{aligned} \varphi_1(x, y) &= \left(\left(-1 + \frac{4C_{XY}C_{YX}}{C_X C_Y} \right) x - \frac{2C_{YX}}{C_Y} y, -y + \frac{2C_{XY}x}{C_X} \right), \\ \varphi_2(x, y) &= \left(-x + \frac{2C_{YX}}{C_Y} y, -\frac{2C_{XY}}{C_X} x + \left(\frac{4C_{XY}C_{YX}}{C_X C_Y} - 1 \right) y \right), \end{aligned}$$

where $t_1(x, y) = -\frac{2x}{C_X}$, $t_2(x, y) = -\frac{2}{C_Y}(-\frac{2C_{XY}}{C_X}x + y)$, $t_3(x, y) = -\frac{2}{C_Y}y$ and $t_4(x, y) = -\frac{2}{C_X}(x - \frac{2C_{YX}}{C_Y}y)$. As $X^2h(0) < 0$ and $Y^2h(0) > 0$ we get $t_1 > 0$ and $t_3 > 0$, respectively.

To prove item (a) we need to solve $\varphi_1(p) = \varphi_2(p)$ with $p = (x, y, 0) \in \Sigma$. Initially we suppose that $p \in EscR$. From (4) and solving $\varphi_1(x, y) = \varphi_2(x, y)$ we obtain:

- (i) $(X(Yh))(0)(Y(Xh))(0) = (X^2h)(0)(Y^2h)(0)$;
- (ii) $(x, y) \in \mathcal{C}_{\lambda^*}$, where $\lambda^* = \frac{X^2h(0)}{XYh(0)}$.

As $\mathcal{C}_{\lambda^*} \subset EscR$ we obtain that $XYh(0) > 0$. Replacing this condition in (i) we get $YXh(0) < 0$.

In this way, we consider the diffeomorphisms φ_1 and φ_2 restricted to $RS \subset \Sigma$. By Lemma 15, $RS = \emptyset$.

Therefore, for $Z \in SL(\Omega_0^F(\delta))$ there does not exist pseudo periodic passing by $p \in EscR$.

If $p \in SwR$ follows by item (c) of Theorem 10 the result. Observe that in this case, some values of the time $t_i, i = 1, \dots, 4$ are equal to 0.

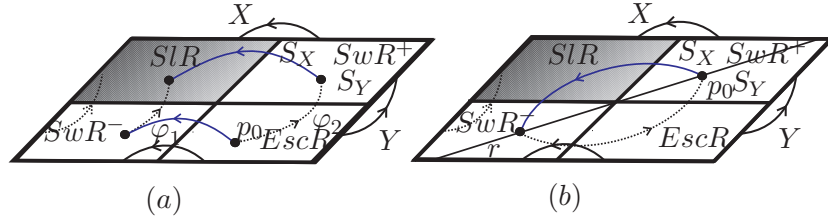


FIGURE 4. In (a) is represented a pseudo periodic orbit. In (b) is represented one of the periodic orbits of $Z \in SL(\Omega)$.

Let us prove item (b). For $Z \in \Omega_0^P(\delta)$, the conditions (i), (ii) and $\varphi_1 = \varphi_2$ are satisfied since $\mathcal{C}_{\lambda^*} \subset SwR$. The dynamics of $Z \in \Omega_0^P(\delta)$ is illustrated in Figure 4.

Claim: “If $p \in \mathcal{C}_{\lambda^*} \subset SwR^+$ and $Z \in \Omega_0^P(\delta)$ (respectively $p \in \mathcal{C}_{\lambda^*} \subset SwR^-$ and $Z \in \Omega_0^P(\delta)$) then $\varphi_1(p) = p$ (respectively $\varphi_2(p) = p$). In other words, on \mathcal{C}_{λ^*} there is a family of periodic orbits”.

In fact, observe that if $Z \in \Omega_0^P(\delta), p \in \mathcal{C}_{\lambda^*} \subset SwR$ then Z satisfies the conditions (i), (ii) and $t_i \geq 0$, for $i = 1, \dots, 4$. That is, $\varphi_1(p) = p$ ($p \in \mathcal{C}_{\lambda^*} \cap SwR^+$) or $\varphi_2(p) = p$ ($p \in \mathcal{C}_{\lambda^*} \cap SwR^-$). Besides, for $Z \in \Omega_0^P(\delta)$ we have

$\mathcal{C}_{\lambda^*} \subset (SwR^+ \cup SwR^-)$. If $p \in [\mathcal{C}_{\lambda^*} \cap SwR^-]$ then $\varphi_1(p) = \phi_Y^{t_2} \circ \phi_X^{t_1}(p) = -p$ and $t_1(x, y) = 0$. ■

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REFERENCES

- [B-S-C] Bizzari F., Storace M. and Colombo A., *Bifurcation analysis of an impact model for forest fire prediction*, Int. J. Bifurcat. Chaos, 18, 22752288, 2008.
- [C] Chillingworth D.R.J., *Discontinuity geometry for an impact oscillator*, Dynam. Syst., 17(4):389420, 2002.
- [C-B-F-J] Colombo A., Bernardo M. di, Fossas E. and Jeffrey M. R., *Teixeira singularities in 3D switched feedback control systems*, Systems and Control Letters, Volume 59, Issue 10, pp. 615-622, October 2010.
- [F] Filippov A. F., *Differential equations with discontinuous righthand sides*, vol. 18 of Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers Group, Dordrecht, 1988.
- [J] Jeffrey M.R., *Two-folds in nonsmooth dynamical systems*, I.F.A.C., Chaos09 proceedings, Queen Mary, Junho de 2009.
- [J-C] Jeffrey M.R. and Colombo A., *The two-fold singularity of discontinuous vector fields*, SIAM J. Appl. Dyn. Syst. Volume 8, Issue 2, pp. 624-640,(2009).
- [K-R-G] Yu. A. Kuznetsov, S. Rinaldi, and A. Gragnani., *One parameter bifurcations in planar Filippov systems*, Int. J. Bifurcat. Chaos, 13, 21572188, 2003.
- [S-T] Sotomayor J. and Teixeira M. A., *Vector fields near the boundary of a 3-manifold*, Lect. Notes in Math., **331**, Springer Verlag, (1988),169-195.
- [T1] Teixeira M. A., *Stability conditions for discontinuous vector fields*, J. Diff. Eq., **V 88**,(1990),15-29.
- [T2] Teixeira M. A., *Perturbation theory for non-smooth systems*, Encyclopedia of Complexity and Systems Science, vol. 22, Springer, 2009.
- [T3] Teixeira M. A., *Generic bifurcations of sliding vector fields*, Journal of Mathematical Analysis and Applications, **176**,(1993),436-457.

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